The problem of track reconstruction in the sweeper magnet

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Abstract

The problem of track reconstruction in the sweeper magnet is manifold. First, it is imperative to get a good measurement of the magnetic field in the mid-plane of the gap to be able to reconstruct the full three-dimensional magnetic field outside of the midplane. This can be acomplished by Fourier transfomations. The relevant magnetostatic equations are discussed. A second problem relates to the target-position dependence of the particle tracks. We propose to use the method of partially inverted ion-optical matrices (which make use of the information from beamline tracking detectors) to better reconstruct tracks of charged fragments from the focal-plane box onto the target. This method can provide a more accurate reconstruction than the fully-inverted ion-optical matrix produced by the program COSY.

1 Introduction

Reconstruction of tracks has been difficult for the sweeper magnet. One reason is that the magnet has a wide gap (> 10 cm) which is the cause for a relatively inhomogenous magnetic field with significant gradients such that non-linear terms in the ion-optical matrix become important. This makes an accurate construction of the three-dimensional magnetic field impostant. Another reason is that the sweeper magnet cannot run in dispersion-matched mode since it lacks focussing elements. Running the sweeper magnet in focused mode is the only option, however, the focus of secondary beams on the target is typically rather large (~ 2 cm). This fact makes the problem of reconstruction much more complicated than for a small-gap magnet with a highly homogenous field like the dipoles used in the A1900 or S800.

In the first part of this work, we discuss how a three-dimensional magnetic field can be constructed by means of a Fourier transformation and only using measurement along the mid-plane. In the second part, we will propose the method of partially inverted ion-optical matrices which will increase the accuracy of reconstruction by making use of tracking information of the incoming beam onto the target position.

2 Magnetic field construction from mid-plane measurements

The problem we address here is obviously a magnetostatic problem, hence the relevant Maxwell equations are

$$\nabla \vec{B} = 0 \quad (1)$$

$$\nabla \times \vec{B} = \mu_0 \vec{\jmath}. \quad (2)$$

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Since we are only concerned with reconstructing the magnetic field inside the gap where no currents are present, Eq. (2) simplifies to

$$\nabla \times \vec{B} = \vec{0}, \tag{3}$$

which has the solution

$$\vec{B} = \nabla \Phi. \tag{4}$$

Inserting this solution into Eq. (1), we obtain a Laplace equation for Φ

$$\triangle \Phi = 0. \tag{5}$$

The solution of the Laplace equation has the form

$$\begin{aligned} \Phi(x,y,z) = \int & \mathrm{d}k_x \mathrm{d}k_y \sinh\left(\sqrt{k_x^2+k_y^2}\, z\right) \\ & [B_1(k_x,k_y)\cos(k_x x)\cos(k_y y) + B_2(k_x,k_y)\cos(k_x x)\sin(k_y y) \\ & + B_3(k_x,k_y)\sin(k_x x)\cos(k_y y) + B_4(k_x,k_y)\sin(k_x x)\sin(k_y y)] \end{aligned} \tag{6}$$

where we have imposed the following boundary conditions: $z = 0$ is the mid-plane inside the magnet gap where for symmetry reasons $B_x = B_y = 0$, and where $B_z \to 0$ for either $x, y \to \infty$. The $B_i(k_x, k_y)$, $i = 1 \ldots 4$ are related to the Fourier components of the mid-plane magnetic field, since it is

$$\begin{aligned} \vec{B}(x,y,z=0) = \vec{e}_z \int & \mathrm{d}k_x \mathrm{d}k_y \sqrt{k_x^2+k_y^2} \\ & [B_1(k_x,k_y)\cos(k_x x)\cos(k_y y) + B_2(k_x,k_y)\cos(k_x x)\sin(k_y y) \\ & + B_3(k_x,k_y)\sin(k_x x)\cos(k_y y) + B_4(k_x,k_y)\sin(k_x x)\sin(k_y y)] , \end{aligned} \tag{7}$$

and thus they are fully determined by the z component of the mid-plane magnetic field using the inverse Fourier transformation

$$B_1(k_x,k_y) = \frac{1}{(2\pi)^2\sqrt{k_x^2+k_y^2}} \int \mathrm{d}x\mathrm{d}y\, B_z(x,y,z=0)\cos(k_x x)\cos(k_y y) \tag{8}$$

$$B_2(k_x,k_y) = \frac{1}{(2\pi)^2\sqrt{k_x^2+k_y^2}} \int \mathrm{d}x\mathrm{d}y\, B_z(x,y,z=0)\cos(k_x x)\sin(k_y y) \tag{9}$$

$$B_3(k_x,k_y) = \frac{1}{(2\pi)^2\sqrt{k_x^2+k_y^2}} \int \mathrm{d}x\mathrm{d}y\, B_z(x,y,z=0)\sin(k_x x)\cos(k_y y) \tag{10}$$

$$B_4(k_x,k_y) = \frac{1}{(2\pi)^2\sqrt{k_x^2+k_y^2}} \int \mathrm{d}x\mathrm{d}y\, B_z(x,y,z=0)\sin(k_x x)\sin(k_y y). \tag{11}$$

Hence it is sufficient to measure only the z component of the mid-plane magnetic field (from $-\infty$ to ∞) to be able to reconstruct inside the magnet gap the full magnetic field vector even off mid plane. Once the magnetic field is fully known, the construction of the ion-optical matrix is simple.

The one remaining problem is that the mid-plane magnetic field has to be extrapolated to distances where it is not measured. The program COSY (which also determines the ion-optical matrix from the reconstructed magnetic field) assumes an iron-dominated fringe field which is typically described by Enge functions. However, we know that due to the presence of the coils, the magnetic field actually changes polarization at some short distance (~ 5 cm) outside the vacuum chamber, such that this assumption is most likely not fulfilled. It is not clear how large a systematic uncertainty into the calculation of the ion-optical matrix is introduced by this fact. One way to check is to look whether the reconstructed magnetic field is in agreement with measurements off mid plane. By using more realistic extrapolations, such an agreement will certainly be improved.

As another application of the off mid plane measurements it has been proposed to fully reconstruct the magnetic field by first interpolating measurements of the z component of the magnetic field at different values of z and then using the equation $\oint \mathrm{d}\vec{r}\vec{B}$ in small steps to reconstruct the corresponding x and y components of the magnetic field. However, this method ignores the condition $\nabla\vec{B} = 0$ and puts in its place a simple (and most likely non physical) interpolation between measurement points. It is even more difficult to estimate the systematic uncertainty of this effect on the track reconstruction.

3 Partial inversion of ion-optical matrices

3.1 Forward ion-optical matrices

An ion-optical matrix relates coordinates $(x, \theta_x, y, \theta_y, \Delta E)^{(T)}$ at a certain position in front of a magnetic element such as a spectrograph ('target position') to coordinates $(x, \theta_x, y, \theta_y, L)^{(D)}$ behind the spectrograph (detector position). In this work, x is the dispersive plane and y is the non-dispersive plane of the spectrograph. L is the tracklength of a particle. If there is no material in the beam, the energy E (or the relative energy difference $\Delta E = (E - E_0)/E_0$ in % from a reference energy E_0) remains unchanged. In first order, coordinates belonging to the dispersive plane and non-dispersive plane in front of and behind the magentic element are not mixed. Hence, a typical first-order ion-optical matrix will look like the following

$$\begin{pmatrix} x \\ \theta_x \\ y \\ \theta_y \\ L \end{pmatrix}^{(D)} = \begin{pmatrix} M_{xx} & M_{x\theta_x} & 0 & 0 & M_{x\Delta E} \\ M_{\theta_x x} & M_{\theta_x\theta_x} & 0 & 0 & M_{\theta_x\Delta E} \\ 0 & 0 & M_{yy} & M_{y\theta_y} & 0 \\ 0 & 0 & M_{\theta_y y} & M_{\theta_y\theta_y} & 0 \\ M_{Lx} & M_{L\theta_x} & 0 & 0 & M_{L\Delta E} \end{pmatrix} \begin{pmatrix} x \\ \theta_x \\ y \\ \theta_y \\ \Delta E \end{pmatrix}^{(T)} \tag{12}$$

where for the two 2×2 submatrices which are mixing position and angle in one direction $\det M = 1$ is required in order to preserve the phase-space volume. The output coordinates can also depend on higher-order combinations of input coordinates. In such a case, the input vector is extended to all possible higher-order combinations and the ion-optical matrix is modified accordingly. In first order, there are only 5 linear input coordinates, hence the ion-optical matrix is a 5×5 matrix. In second order, there are 15 new quadratic input coordinates, the ion-optical matrix has to be extended by a 5×15 matrix. In third order, there are 35 new third-order input coordinates and the ion-optical matrix has to be extended by a 35×5 matrix etc.

3.2 Fully inverse ion-optical matrices

In typical spectrographic applications, the position and angle of particles at the detector position are used to determine the energy, angle, and position of those particles at the target position. For this reason, the ion-optical matrix has to be inverted. However, looking at Eq. (12), we notice that a full inversion is not possible, since we only know 4 out of 5 coordinates of the particle behind the spectrograph (we do not know the tracklength). This fundamental problem has been solved in two different ways. (i) if sufficient magnetic elements are available, the spectrograph can be run in dispersion-matched mode, in which essentially the matrix elements M relating the input coordinate $x^{(T)}$ to any of the output coordinates are negligibly small, or (ii) the spectrograph can be run in focused mode, in which the incoming beam is focused in the dispersive plane at the point of the input coordinate, such that one can assume that the input coordinate $x^{(T)} = 0$. Either way, Eq. (12) can effectively be reduced to

$$\begin{pmatrix} x \\ \theta_x \\ y \\ \theta_y \end{pmatrix}^{(D)} = \begin{pmatrix} M_{x\theta_x} & 0 & 0 & M_{x\Delta E} \\ M_{\theta_x\theta_x} & 0 & 0 & M_{\theta_x\Delta E} \\ 0 & M_{yy} & M_{y\theta_y} & 0 \\ 0 & M_{\theta_y y} & M_{\theta_y\theta_y} & 0 \end{pmatrix} \begin{pmatrix} \theta_x \\ y \\ \theta_y \\ \Delta E \end{pmatrix}^{(T)} \tag{13}$$

where the column concerning the target coordinate $x^{(T)}$ and the row concerning the tracklength L have been eliminated. After inverting this matrix, the coordinates $(\theta_x, y, \theta_y, E)^{(D)}$ can be reconstructed from position and angle measurements at the detector position. This is the kind of inverse ion-optical matrix which is provided by the program COSY. In this work, we will call these matrices 'fully inverse ion-optical matrices'.

Unfortunately, the sweeper magnet by itself cannot be run in dispersion-matched mode, since it lacks additional magnetic elements. Moreover, the incoming beam, since it is a secondary beam, is typically of poor quality; the focus in the dispersive plane is never better than ~ 2 cm, hence, the setup does not lend itself to be run in focused mode. For this reason, we propose a different solution to the above problem, making use of the beamline tracking detectors.

3.3 Partial inverse ion-optical matrices

With the help of beamline tracking detectors, the $x^{(T)}$ and $y^{(T)}$ target positions are known and can be used to improve the energy reconstruction (the angles at target position are determined by the nuclear reaction in the target, they cannot be infered by beamline tracking detectors). Actually, since we now have two additional pieces of information, the problem of reconstruction becomes overdetermined. Therefore, we will only use the $x^{(T)}$ target position as additional input, the $y^{(T)}$ target position will be used only in comparison with the reconstructed $y^{(T)}$ target position as a check. In the following, we will provide a mechanism by which the additional information on the $x^{(T)}$ target position is entered into the reconstruction.

To explain the principle behind this method, we look at the part of the first-order ion-optical matrix which relates only the coordinates in the dispersive plane

$$\begin{pmatrix} x \\ \theta_x \\ L \end{pmatrix}^{(D)} = \begin{pmatrix} M_{xx} & M_{x\theta_x} & M_{x\Delta E} \\ M_{\theta_x x} & M_{\theta_x\theta_x} & M_{\theta_x\Delta E} \\ M_{Lx} & M_{L\theta_x} & M_{L\Delta E} \end{pmatrix} \begin{pmatrix} x \\ \theta_x \\ \Delta E \end{pmatrix}^{(T)} . \tag{14}$$

Here, we have target $x^{(T)}$ as input coordinate on the right-hand side and detector $x^{(D)}$ and $\theta_x^{(D)}$ as output coordinates on the left-hand side. All three of which are known quantities. The unknown quantities are target $\theta_x^{(T)}$ and tracklength L on the left-hand side and ΔE on the right-hand side. It is now straightforward to exchange a coordinate on the left-hand side with one coordinate on the right-hand side by using the following rules:

- the 'pivot', i.e., the matrix element which relates the two coordinates which are to be exchanged transforms as $a \to 1/a$
- elements in the same row as the 'pivot' transform as $b \to -b/a$
- elements in the same column as the 'pivot' transform as $c \to c/a$

- all other elements transform as $d \to d - cb/a$.

Hence, if we want to exchange, say, ΔE with detector $x^{(D)}$, the ion-optical matrix will transform as

$$\begin{pmatrix} \Delta E \\ \theta_x^{(D)} \\ L \end{pmatrix} = \begin{pmatrix} -M_{xx}/M_{x\Delta E} & -M_{x\theta_x}/M_{x\Delta E} & 1/M_{x\Delta E} \\ M_{\theta_x x} - M_{xx}M_{\theta_x \Delta E}/M_{x\Delta E} & M_{\theta_x\theta_x} - M_{x\theta_x}M_{\theta_x\Delta E}/M_{x\Delta E} & M_{\theta_x\Delta E}/M_{x\Delta E} \\ M_{Lx} - M_{xx}M_{L\Delta E}/M_{x\Delta E} & M_{L\theta_x} - M_{x\theta_x}M_{L\Delta E}/M_{x\Delta E} & M_{L\Delta E}/M_{x\Delta E} \end{pmatrix} \begin{pmatrix} x^{(T)} \\ \theta_x^{(T)} \\ x^{(D)} \end{pmatrix}. \tag{15}$$

In a second step, one can now exchange target $\theta_x^{(T)}$ and detector $\theta_x^{(D)}$, such that the three known quantities are on the right-hand side and the three unknown quantities are on the left-hand side. Ion-optical matrices, where coordinates have been exchanged in this fashion are called 'partial inverse ion-optical matrices' in this work.

It is straightforward to generalize this method to the 5×5 matrices of first order. At this point, we would like to make three comments. (i) exchanges of two coordinates are only possible if the 'pivot' does not equal zero, i.e., it is only possible to exchange coordinates in the dispersive plane and in the non-dispersive plane among themselves. (ii) numerically, the method works better the larger the 'pivot' element is chosen. Ideally, the 'pivot' element should be chosen such that it is larger than one. (iii) when exchanging all five coordinates along the diagonal, the resulting ion-optical matrix is the inverse of the original ion-optical matrix, i.e., in first order, the result of the method is mathematically exact.

The pertinent rules for exchanging coordinates in second or higher order can be derived from the following considerations. Assume the input, i.e., target coordinates of a forward ion-optical matrix are x_i, the output, i.e., detector coordinates are y_j with $i, j = 1 \ldots 5$. The coefficients of the ion-optical matrix are $a^{(j)}_{n_1,n_2,n_3,n_4,n_5}$, such that

$$y_{j=1\ldots5} = \sum_{1 \le n_1+n_2+n_3+n_4+n_5 \le N} a^{(j=1\ldots5)}_{n_1,n_2,n_3,n_4,n_5} x_1^{n_1} x_2^{n_2} x_3^{n_3} x_4^{n_4} x_5^{n_5} \tag{16}$$

where N is the order of the ion-optical matrix. Without loss of generality, we discuss only the exchange of the target coordinate x_5 with the detector coordinate y_5. We are therefore interested in the matrix $b^{(j)}_{n_1,n_2,n_3,n_4,n_5}$ which relates the new input coordinates x_i, y_5 with $i = 1 \ldots 4$ to the new output coordinates y_j, x_5 with $j = 1 \ldots 4$

$$y_{j=1\ldots4} = \sum_{1 \le m_1+m_2+m_3+m_4+m_5 \le N} b^{(j=1\ldots4)}_{m_1,m_2,m_3,m_4,m_5} x_1^{m_1} x_2^{m_2} x_3^{m_3} x_4^{m_4} y_5^{m_5} \tag{17}$$

$$x_5 = \sum_{1 \le m_1+m_2+m_3+m_4+m_5 \le N} b^{(5)}_{m_1,m_2,m_3,m_4,m_5} x_1^{m_1} x_2^{m_2} x_3^{m_3} x_4^{m_4} y_5^{m_5}. \tag{18}$$

In the first step, Eq. (16) is entered into Eq. (18) and, order by order, a system of M linear equations can be constructed from which the coefficients $b^{(5)}$ are determined by simple matrix inversion. In the second step, Eq. (18) is entered into Eq. (16) and by a comparison with Eq. (17) in a similar fashion as in the first step, the coefficients $b^{(1\ldots4)}$ can be deduced. Table 1 shows the dimension M of the system of linear equations which has to be solved when exchanging one coordinate as function of the order of the ion-optical matrix (assuming five input and output variables in the ion-optical matrix).

This algorithm is encoded in the FORTRAN program 'inverse.f' which exchanges coordinates in ion-optical matrices of five input and output variables up to third order. The procedure can be generalized to higher than third order, however, such a generalization will be tedious work. In contrast to the first-order inversion, exchange of coordinates of higher-order ion-optical matrices is only exact up to this order, i.e., when exchanging all five coordinates along the diagonal, the resulting ion-optical matrix is the inverse of the original one only up to the order N. Higher-order terms will not cancel out.

order N	1	2	3	4	5
dimension M	5×5	15×15	35×35	70×70	126×126

Table 1: Dimension M of the system of linear equations which has to be solved when exchanging one coordinate. M is a function of N, which is the order of the ion-optical matrix, assuming five input and output variables.

4 How to produce a partial inverse ion-optical matrix

As a starting point, the forward ion-optical matrix is assumed to have the following output (detector) and input (target) vectors

$$\begin{pmatrix} x^{(D)} \\ \theta_x^{(D)} \\ y^{(D)} \\ \theta_y^{(D)} \\ L \end{pmatrix}_{\text{output}} = \mathcal{M} \begin{pmatrix} x^{(T)} \\ \theta_x^{(T)} \\ y^{(T)} \\ \theta_y^{(T)} \\ \Delta E \end{pmatrix}_{\text{input}} . \quad (19)$$

The goal is to exchange the four known quantities in the output vector against the four unknown quantities in the input vector such that we produce a partial inverse ion-optical matrix of the form

$$\begin{pmatrix} \theta_x^{(T)} \\ y^{(T)} \\ \theta_y^{(T)} \\ \Delta E \\ L \end{pmatrix}_{\text{output}} = \mathcal{M} \begin{pmatrix} x^{(D)} \\ \theta_x^{(D)} \\ y^{(D)} \\ \theta_y^{(D)} \\ x^{(T)} \end{pmatrix}_{\text{input}} . \quad (20)$$

In the first step, let us, e.g., exchange $y^{(T)}$ and $y^{(D)}$. However, we also ask the program to put $y^{(T)}$ at the second position in the new output vector (moving $\theta_x^{(D)}$ to the third position), hence after the first step, the new matrix has the following input and output vectors

$$\begin{pmatrix} x^{(D)} \\ y^{(T)} \\ \theta_x^{(D)} \\ \theta_y^{(D)} \\ L \end{pmatrix}_{\text{output}} = \mathcal{M} \begin{pmatrix} x^{(T)} \\ \theta_x^{(T)} \\ y^{(D)} \\ \theta_y^{(T)} \\ \Delta E \end{pmatrix}_{\text{input}} . \quad (21)$$

In the second step, we finish up with the coordinates in the non-dispersive plane by exchanging $\theta_y^{(T)}$ and $\theta_y^{(D)}$. We ask the program to place $\theta_y^{(T)}$ on third position in the new output vector (moving $\theta_x^{(D)}$ even further down to fourth position) and obtain

$$\begin{pmatrix} x^{(D)} \\ y^{(T)} \\ \theta_y^{(T)} \\ \theta_x^{(D)} \\ L \end{pmatrix}_{\text{output}} = \mathcal{M} \begin{pmatrix} x^{(T)} \\ \theta_x^{(T)} \\ y^{(D)} \\ \theta_y^{(D)} \\ \Delta E \end{pmatrix}_{\text{input}} . \quad (22)$$

In the third step, we have to exchange coordinates in the dispersive plane, e.g., $\theta_x^{(T)}$ and $x^{(D)}$. Now, we ask the program to place $x^{(D)}$ at the first position of the new input vector (moving $x^{(T)}$ down to second position) and obtain

$$\begin{pmatrix} \theta_x^{(T)} \\ y^{(T)} \\ \theta_y^{(T)} \\ \theta_x^{(D)} \\ L \end{pmatrix}_{\text{output}} = \mathcal{M} \begin{pmatrix} x^{(D)} \\ x^{(T)} \\ y^{(D)} \\ \theta_y^{(D)} \\ \Delta E \end{pmatrix}_{\text{input}} . \tag{23}$$

In the final step, we have to exchange the remaining unknown coordinate in the input vector ΔE against the remaining known coordinate in the output vector $\theta_x^{(D)}$. We ask the program to place $\theta_x^{(D)}$ on second position of the new input vector (which automatically moves $x^{(T)}$ down to fifth position) and obtain the desired

$$\begin{pmatrix} \theta_x^{(T)} \\ y^{(T)} \\ \theta_y^{(T)} \\ \Delta E \\ L \end{pmatrix}_{\text{output}} = \mathcal{M} \begin{pmatrix} x^{(D)} \\ \theta_x^{(D)} \\ y^{(D)} \\ \theta_y^{(D)} \\ x^{(T)} \end{pmatrix}_{\text{input}} . \tag{24}$$

It is always possible to obtain this format (the 'standard' format) after four exchanges of coordinates by requiring within the program specific positions for the exchanged coordinates in the input and output vectors. The final result can be checked against the fully inverted ion-optical matrix provided by COSY, since reducing the matrix by the fifth row and the fifth column should produce the same result as COSY coefficient by coefficient. It is important to note that the first exchange of two coordinates in the dispersive as well as the non-dispersive plane should be done for a pair which shares a large (ideally larger than one) first-order matrix element (the 'pivot'). This is not important for the exchange of the second coordinate in either plane, since the pair which has to be exchanged then has already been determined by the previous choice.

5 Other applications

The quadrupole doublet bewteen the two beamline tracking detectors is currently unused. However, a partial inverse ion-optical matrix (where we neglect any chromatic terms) could provide a relation between the four position coordinates as inputs and the four angle coordinates as outputs.

Another possibility is the use of the time-of flight information together with the position information to deduce the momentum of incoming beam particles event by event.

6 Comparison to other methods

Other methods to improve reconstruction have been proposed. First, it has been proposed to produce an array of ion-optical matrices with different starting points in energy, and position and angle in the dispersive plane (the non-dispersive plane does not lend itself to this method, since the program COSY can only provide central trajectories in the mid-plane of the magnetic field). The problem here is that when this array of ion-optical matrices is inverted, it is not possible to say which one of the inverted ion-optical matrices should be applied on an event by event basis, since for that, one would need a priori knowledge of the energy or the angle in the dispersive plane at the target position. The most clear-cut application would be to create an array of ion-optical matrices with different starting points in dispersive plane position. From the beamline tracking, it would be easy to chose the appropriate inverse ion-optical matrix on an event by event basis. However, the method of partial inverse ion-optical matrices seems to make this approach superfluous. On the other hand, the two methods could be combined, such that an array of partial inverse ion-optical matrices could be created, where the correct one to be applied on an event-by-event basis might be identified using consistency checks or an iterative procedure. However, this seem quite cumbersome.

A further complication which has been encountered is that producing ion-optical matrices for different central trajectories will produce different results for the forward tracking of the same particle. This points to that even the lowest non-linear orders of the ion-optical matrix are not sufficient to correctly describe events in the whole available phase space. It is unclear why this is the case. A method was developed by Kazuo Ieki in which a neural network is trained with an array of forward ion-optical matrices (to capture the correct dynamics within the phase space pertaining to the dispersive plane). However, the problem still remains that the program COSY only produces central trajectories within the mid plane of the magnetic field. Any dynamics in the non-dispersive plane therefore rests on the extrapolation of the magnetic field as described in Sect. 2 and on the correct description of the phase space dynamics in the non-dispersive plane by the lowest non-linear orders of the ion-optical matrices.